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A gauge-invariant Hamiltonian description of the motion of charged test particles

Dariusz Chruściński [a], Jerzy Kijowski [b,*]

[a] *Institute of Physics, Nicholas Copernicus University, ul. Grudziądzka 5/7, 87-100 Toruń, Poland*

[b] *Centrum Fizyki Teoretycznej PAN, Aleja Lotników 32/46, 02-668 Warsaw, Poland*

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Abstract

New, gauge-independent, second-order Lagrangian for the motion of classical, charged test particles is used to derive the corresponding Hamiltonian formulation. For this purpose a (relatively little known) Hamiltonian description of theories derived from second-order Lagrangians is presented. Unlike in the standard approach, the canonical momenta arising here are explicitly gauge-invariant and have a clear physical interpretation. The reduced symplectic form obtained this way is (almost) equivalent to Souriau's form. This approach illustrates a new method of deriving equations of motion from field equations. © 1998 Published by Elsevier Science B.V.

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1. Introduction

In [1] a new method of deriving equations of motion from field equations was proposed. The method is based on a new idea of renormalization in classical field theory and a deep analysis of the geometric structure of generators of its symmetry group. It may be applied to any special-relativistic, Lagrangian field theory. When applied to electrodynamics, it leads uniquely to a manifestly gauge-invariant, second-order Lagrangian $\mathcal{L}$ for the motion of charged test particles:

$$\mathcal{L} = L_{\text{particle}} + \mathcal{L}_{\text{int}} = -\sqrt{1-\mathbf{v}^2}\,(m - a^\mu u^\nu M^{\text{int}}_{\mu\nu}(t, \mathbf{q}, \mathbf{v})), \tag{1}$$

* Corresponding author.

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where u^μ denotes the (normalized) four-velocity vector

$$(u^\mu) = (u^0, u^k) := \frac{1}{\sqrt{1-\mathbf{v}^2}}(1, v^k), \tag{2}$$

and $a^\mu := u^\nu \nabla_\nu u^\mu$ is the particle's acceleration (we use the Heaviside–Lorentz system of units with the velocity of light $c = 1$). The skew-symmetric tensor $M^{\text{int}}_{\mu\nu}(t, \mathbf{q}, \mathbf{v})$ is equal to the amount of the angular-momentum of the field, which is acquired by our physical system, when the Coulomb field $\mathbf{f}^{(y,\mathbf{v})}_{\mu\nu}$ accompanying the particle moving with constant velocity $\mathbf{v}$ through the space–time point $y = (t, \mathbf{q})$ is added to the background (external) field. More precisely, the total energy–momentum tensor corresponding to the sum of the background field $f_{\mu\nu}$ and the above Coulomb field decomposes in a natural way into a sum of (1) terms quadratic in the background field, (2) terms quadratic in the Coulomb field, and (3) mixed terms describing their interaction. The quantity $M^{\text{int}}_{\mu\nu}$ is equal to this part of the total angular-momentum $M_{\mu\nu}$, which we obtain integrating only the mixed terms of the energy–momentum tensor (see Section 3 for detailed discussion).

The above result is a by-product of a consistent theory of interacting particles and fields (cf. [2,3]), called *electrodynamics of moving particles.*

We have proved in [1] that the new Lagrangian (1) differs from the standard one,

$$L = L_{\text{particle}} + L_{\text{int}} = -\sqrt{1-\mathbf{v}^2}\,(m - eu^\mu A_\mu(t, \mathbf{q})), \tag{3}$$

by (gauge-dependent) boundary corrections only. We have a similar situation in general relativity, which may be derived either from the gauge-invariant, second-order Hilbert Lagrangian or – equivalently – from the first-order, coordinate-dependent Einstein Lagrangian, quadratic with respect to the connection coefficients.

Both Lagrangians generate, therefore, the same equations of motion for test particles in a given external field. In the present paper we explicitly derive these equations and construct the gauge-invariant Hamiltonian version of this theory.

Standard Hamiltonian formalism, based on the gauge-dependent Lagrangian (3), leads to the gauge-dependent Hamiltonian

$$H(t, \mathbf{q}, \mathbf{p}) = \sqrt{m^2 + (\mathbf{p} + e\mathbf{A}(t, \mathbf{q}))^2} + eA_0(t, \mathbf{q}), \tag{4}$$

and the gauge-dependent momentum

$$p_k := p^{\text{kin}}_k - eA_k(t, \mathbf{q}) = mu_k - eA_k(t, \mathbf{q}) \tag{5}$$

canonically conjugate to the particle's position q^k. This gauge-dependence leads to serious conceptual difficulties, if we want to describe quantized particles in a time-dependent field (e.g. a plane wave), and have no privileged gauge (e.g. time-independent) at our disposal.

As was observed by Souriau (see [5]), we may replace the non-physical momentum (5) in the description of the phase space of this theory by the gauge-invariant quantity p^{kin}.

The price we pay for this change is that the canonical pre-symplectic form, corresponding to the theory of free particles

$$\Omega = \mathrm{d}p^{\mathrm{kin}}_{\mu} \wedge \mathrm{d}q^{\mu}, \tag{6}$$

has to be replaced by its deformation

$$\Omega_S := \Omega - e\, f_{\mu\nu}\, \mathrm{d}q^{\mu} \wedge \mathrm{d}q^{\nu}, \tag{7}$$

where e is the particle's charge.

Both Ω and Ω_S are defined on the "mass-shell" of the kinetic momentum, i.e. on the surface $(p^{\mathrm{kin}})^2 = -m^2$ in the cotangent bundle T^*M over the space–time M (we use the Minkowskian metric with the signature $(-, +, +, +)$). These forms contain the entire information about dynamics: for free particles the admissible trajectories are those, whose tangent vectors belong to the degeneracy distribution of Ω. Souriau noticed that replacing (6) by its deformation (7) we obtain the theory of motion of the particle in a given electromagnetic field $f_{\mu\nu}$.

The new approach, proposed in the present paper, is based on Lagrangian (1). It leads *directly* to a perfectly gauge-invariant Hamiltonian, having a clear physical interpretation as the sum of two terms: (1) kinetic energy mu_0 and (2) "interaction energy" equal to the amount of field energy acquired by our physical system, when the particle's Coulomb field is added to the background field.

When formulated in terms of pre-symplectic geometry, our approach leads uniquely to a new form Ω_N:

$$\Omega_N := \Omega - e\, h_{\mu\nu}\, \mathrm{d}q^{\mu} \wedge \mathrm{d}q^{\nu}\,, \tag{8}$$

where

$$h_{\mu\nu} := 2(f_{\mu\nu} - u_{[\mu} f_{\nu]\lambda} u^{\lambda}) \tag{9}$$

(brackets denote antisymmetrization), i.e. we prove the following:

Theorem 1. *One-dimensional submanifolds of the particle's mass shell, whose tangent vectors belong to the degeneracy distribution of the form Ω_N restricted to the shell, are precisely the trajectories of test particles moving in the external electromagnetic field $f_{\mu\nu}$.*

It is easy to see that both Ω_S and Ω_N, although different, have the same degeneracy vectors, because h and f give the same value on the velocity vector u_ν:

$$u^{\nu} h_{\mu\nu} = u^{\nu} f_{\mu\nu}. \tag{10}$$

Hence, both define the same equations of motion. We stress, however, that our Ω_N is *uniquely obtained* from the gauge-invariant Lagrangian (1) *via* the Legendre transformation.

The paper is organized as follows. In Section 2 we sketch briefly the (relatively little known) Hamiltonian formulation of theories arising from the second-order Lagrangian. In Section 3 we prove explicitly that the Euler–Lagrange equations derived from $\mathcal{L}$ are equivalent to the Lorentz equations of motion. Finally, Section 4 contains the gauge-invariant Hamiltonian structure of the theory. The most involved proofs are shifted to Appendix A.

2. Canonical formalism for a second-order Lagrangian theory

Consider a theory described by the second-order Lagrangian $L = L(q^i, \dot{q}^i, \ddot{q}^i)$ (to simplify the notation we will skip the index "i" corresponding to different degrees of freedom q^i; extension of this approach to higher-order Lagrangians is straightforward). Introducing auxiliary variables $v = \dot{q}$ we can treat our theory as a first-order one with Lagrangian constraints $\phi := \dot{q} - v = 0$ on the space of Lagrangian variables $(q, \dot{q}, v, \dot{v})$. Dynamics is generated by the following symplectic relation:

$$\mathrm{d}L(q, v, \dot{v}) = \frac{\mathrm{d}}{\mathrm{d}t}(p\,\mathrm{d}q + \pi\,\mathrm{d}v) = \dot{p}\,\mathrm{d}q + p\,\mathrm{d}\dot{q} + \dot{\pi}\,\mathrm{d}v + \pi\,\mathrm{d}\dot{v}, \tag{11}$$

where (p, π) are momenta canonically conjugate to q and v, respectively. Because L is defined only on the constraint submanifold, its derivative $\mathrm{d}L$ is not uniquely defined and has to be understood as a collection of *all the covectors* which are compatible with the derivative of the function along constraints (cf. [6]). This means that the left-hand side is defined up to $\mu(\dot{q} - v)$, where μ are Lagrange multipliers corresponding to constraints $\phi = 0$. We conclude that $p = \mu$ is arbitrary and (11) is equivalent to the system of dynamical equations:

$$\pi = \frac{\partial L}{\partial \dot{v}}, \tag{12}$$

$$\dot{p} = \frac{\partial L}{\partial q}, \tag{13}$$

$$\dot{\pi} = \frac{\partial L}{\partial v} - p. \tag{14}$$

The last equation implies the definition of the canonical momentum p:

$$p = \frac{\partial L}{\partial v} - \dot{\pi} = \frac{\partial L}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right). \tag{15}$$

Its time derivative

$$\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \dot{v}}\right) \tag{16}$$

is equivalent, due to the second canonical equation (13), to the Euler–Lagrange equation:

$$\frac{\delta L}{\delta q} := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right) + \frac{\partial L}{\partial q} = 0. \tag{17}$$

To obtain Hamiltonian description (see e.g. [4]) we simply apply the Legendre transformation to formula (11):

$$-\,\mathrm{d}H = \dot{p}\,\mathrm{d}q - \dot{q}\,\mathrm{d}p + \dot{\pi}\,\mathrm{d}v - \dot{v}\,\mathrm{d}\pi, \tag{18}$$

where $H(q, p, v, \pi) = p\,v + \pi\,\dot{v} - L(q, v, \dot{v})$. In this formula we have to insert $\dot{v} = \dot{v}(q, v, \pi)$, calculated from Eq. (12). Let us observe that H obtained this way is linear with respect to the momentum p. This is a characteristic feature of the Hamiltonians obtained from a second-order theory.

In generic situation, Euler–Lagrange equations (17) are of the fourth-order. The corresponding four Hamiltonian equations describe, therefore, the evolution of q and its derivatives up to third-order. Due to Hamiltonian equations implied by symplectic relation (18), the information about successive derivatives of q is carried by (v, π, p):

– v describes $\dot{q}$:

$$\dot{q} = \frac{\partial H}{\partial p} \equiv v, \tag{19}$$

and the constraint $\phi = 0$ is reproduced due to linearity of H with respect to p,

– π contains information about $\ddot{q}$:

$$\dot{v} = \frac{\partial H}{\partial \pi}, \tag{20}$$

– p contains information about $\dddot{q}$:

$$\dot{\pi} = -\frac{\partial H}{\partial v} = \frac{\partial L}{\partial v} - p, \tag{21}$$

– the true dynamical equation reads:

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{\partial L}{\partial q}. \tag{22}$$

3. Equations of motion from the variational principle

In this section we explicitly derive the particle's equations of motion from the variational principle based on the gauge-invariant Lagrangian (1). The Euler–Lagrange equations for a second-order Lagrangian theory are given by

$$\dot{p}_k = \frac{\partial \mathcal{L}}{\partial q^k}, \tag{23}$$

where, as we have seen in the previous section, the momentum p_k canonically conjugate to the particle's position q^k is defined as

$$p_k := \frac{\partial \mathcal{L}}{\partial v^k} - \dot{\pi}_k \tag{24}$$

and

$$\pi_k := \frac{\partial \mathcal{L}}{\partial \dot{v}^k} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\, u^\nu M^{\mathrm{int}}_{k\nu}(t, \mathbf{q}, \mathbf{v}). \tag{25}$$

Now,

$$u^\nu M^{\mathrm{int}}_{k\nu} = u^0 M^{\mathrm{int}}_{k0} + u^l M^{\mathrm{int}}_{kl} = -u^0 r^{\mathrm{int}}_k + u^l \epsilon^m_{kl} s^{\mathrm{int}}_m, \tag{26}$$

where $r_k^{\rm int}$ and $s_m^{\rm int}$ are the static momentum and the angular-momentum of the interaction tensor. They are defined as follows: we consider the sum of the (given) background field $f_{\mu\nu}$ and the boosted Coulomb field $\mathbf{f}_{\mu\nu}^{(y,u)}$ accompanying the particle moving with constant four-velocity u and passing through the space–time point $y = (t, \mathbf{q})$. Being bi-linear in fields, the energy–momentum tensor $T^{\rm total}$ of the total field

$$f_{\mu\nu}^{\rm total} := f_{\mu\nu} + \mathbf{f}_{\mu\nu}^{(y,u)} \tag{27}$$

may be decomposed into three terms: the energy–momentum tensor of the background field $T^{\rm field}$, the Coulomb energy–momentum tensor $T^{\rm particle}$, which is composed of terms quadratic in $\mathbf{f}_{\mu\nu}^{(y,u)}$ and the "interaction tensor" $T^{\rm int}$, containing mixed terms:

$$T^{\rm total} = T^{\rm field} + T^{\rm particle} + T^{\rm int} . \tag{28}$$

Interaction quantities (labelled with "int") are those obtained by integrating appropriate components of $T^{\rm int}$. Because all the three tensors are conserved outside of the sources (i.e. outside of two trajectories: the actual trajectory of our particle and the straight line passing through the space–time point y with four-velocity u), the integration gives the same result when performed over *any* asymptotically flat Cauchy 3-surface passing through y.

In particular, $r^{\rm int}$ and $s^{\rm int}$ may be written in terms of the laboratory-frame components of the electric and magnetic fields as follows:

$$r_k^{\rm int}(t, \mathbf{q}, \mathbf{v}) = \int_\Sigma {\rm d}^3x\ (x_k - q_k)(\mathbf{D}\mathbf{D}_0 + \mathbf{B}\mathbf{B}_0), \tag{29}$$

$$s_m^{\rm int}(t, \mathbf{q}, \mathbf{v}) = \epsilon_{mij} \int_\Sigma {\rm d}^3x\ (x^i - q^i)(\mathbf{D}\times\mathbf{B}_0 + \mathbf{D}_0\times\mathbf{B})^j, \tag{30}$$

where $\mathbf{D}$ and $\mathbf{B}$ are components of the external field f, whereas $\mathbf{D}_0$ and $\mathbf{B}_0$ are components of $\mathbf{f}^{(y,u)}$, i.e.

$$\mathbf{D}_0(\mathbf{x}; \mathbf{q}, \mathbf{v}) = \frac{e}{4\pi|\mathbf{x}-\mathbf{q}|^3} \frac{1-\mathbf{v}^2}{(1-\mathbf{v}^2 + (\mathbf{v}(\mathbf{x}-\mathbf{q})/|\mathbf{x}-\mathbf{q}|)^2)^{3/2}} (\mathbf{x}-\mathbf{q}), \tag{31}$$

$$\mathbf{B}_0(\mathbf{x}; \mathbf{q}, \mathbf{v}) = \mathbf{v}\times\mathbf{D}_0(\mathbf{x}; \mathbf{q}, \mathbf{v}). \tag{32}$$

It may be easily seen that quantities $r_k^{\rm int}$ and $s_m^{\rm int}$ are not independent. They fulfill the following condition:

$$s_k^{\rm int} = -\epsilon_{kl}^{m} v^l r_m^{\rm int} . \tag{33}$$

To prove this relation let us observe that in the particle's rest-frame (see Appendix A for the definition) the angular-momentum corresponding to $T^{\rm int}$ vanishes (cf. [1]). When translated to the language of the laboratory frame, this is precisely equivalent to the above relation.

Inserting (33) into (26) we finally get

$$\pi_k = -\left(\delta_k^l + \frac{v^l v_k}{1-\mathbf{v}^2}\right) r_l^{\rm int} . \tag{34}$$

The quantity r_k^{int} depends upon time *via* the time dependence of the external fields $(\mathbf{D}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x}))$, the particle's position $\mathbf{q}$ and the particle's velocity $\mathbf{v}$, contained in formulae (31) and (32) for the particle's Coulomb field.

Now, we are ready to compute p_k from (24):

$$p_k = \frac{m v_k}{\sqrt{1-\mathbf{v}^2}} + \dot{v}^l \frac{\partial \pi_l}{\partial v^k} - \left(\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} + \dot{v}^l \frac{\partial \pi_k}{\partial v^l} \right)$$

$$= p_k^{\mathrm{kin}} - \left(\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} \right) - \dot{v}^l \left(\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} \right). \tag{35}$$

Observe, that the momentum p_k depends upon time, particle's position and velocity but also on particle's acceleration. Hovewer, using (29) one easily shows (see Appendix A):

Lemma 2.

$$\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} = 0. \tag{36}$$

Hence, the term proportional to $\dot{v}^l$ vanishes. Moreover, the following lemma can be proved (see also Appendix A):

Lemma 3.

$$\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} = -p_k^{\mathrm{int}}, \tag{37}$$

where we denote

$$p_k^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}) = \int_{\Sigma} \mathrm{d}^3 x \, (\mathbf{D} \times \mathbf{B}_0 + \mathbf{D}_0 \times \mathbf{B})_k . \tag{38}$$

We see that p_k^{int} is the spatial part of the "interaction momentum":

$$p_\mu^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}) = \int_{\Sigma} T_{\mu\nu}^{\mathrm{int}} \, \mathrm{d}\Sigma^\nu, \tag{39}$$

where Σ is any hypersurface intersecting the particle's trajectory at the point $(t, \mathbf{q}(t))$. The above integral is finite (cf. [2]) and invariant with respect to changes of Σ, provided the intersection point with the trajectory does not change. It was shown in [1] that p_μ^{int} is orthogonal to the particle's four-velocity, i.e. $p_\mu^{\mathrm{int}} u^\mu = 0$.

Finally, the momentum canonically conjugate to the particle's position equals:

$$p_k = p_k^{\mathrm{kin}} + p_k^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}). \tag{40}$$

It is a sum of two terms: kinetic momentum p_k^{kin} and the amount of momentum p_k^{int} which is acquired by our system, when the particle's Coulomb field is added to the background (external) field. We stress that contrary to the standard formulation based on (3), our canonical momentum (40) is gauge-invariant. Now, Euler–Lagrange equations (23) read

$$\frac{\mathrm{d}p_k^{\mathrm{kin}}}{\mathrm{d}t} + \frac{\mathrm{d}p_k^{\mathrm{int}}}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial q^k}, \tag{41}$$

or in a more transparent way:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m v_k}{\sqrt{1-\mathbf{v}^2}}\right) = -\left(\frac{\partial p_k^{\mathrm{int}}}{\partial t} + v^l \frac{\partial p_k^{\mathrm{int}}}{\partial q^l}\right) - \dot{v}^l \left(\frac{\partial p_k^{\mathrm{int}}}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k}\right). \tag{42}$$

Again, using definitions of π_l and p_k^{int} one shows the following:

Lemma 4.

$$\frac{\partial p_k^{\mathrm{int}}}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k} = 0. \tag{43}$$

(For the proof see Appendix A). Hence, the term proportional to the particle's acceleration vanishes. In Appendix A we show that the following identities hold:

Lemma 5.

$$\begin{aligned}\frac{\partial p_k^{\mathrm{int}}}{\partial t} + v^l \frac{\partial p_k^{\mathrm{int}}}{\partial q^l} &= -e\sqrt{1-\mathbf{v}^2}\, u^\nu f_{k\nu}(t,\mathbf{q}) \\ &= -e(E_k(t,\mathbf{q}) + \epsilon_{klm} v^l B^m(t,\mathbf{q})).\end{aligned} \tag{44}$$

We conclude that the Euler–Lagrange equations (23) for the variational problem based on $\mathcal{L}$ are equivalent to the Lorentz equations for the motion of charged particles:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m v_k}{\sqrt{1-\mathbf{v}^2}}\right) = e(E_k(t,\mathbf{q}) + \epsilon_{klm} v^l B^m(t,\mathbf{q})). \tag{45}$$

4. Hamiltonian formulation

By Hamiltonian formulation of the theory we understand, usually, the phase space $\mathcal{P}$ of Hamiltonian variables (q^k, p_k) endowed with the symplectic 2-form $\omega = \mathrm{d}p_k \wedge \mathrm{d}q^k$ and the Hamiltonian function H (the energy of the system) defined on $\mathcal{P}$. However, for time-dependent systems it is more convenient to replace this framework by the so-called *homogeneous* formulation. For this purpose we consider the *evolution space* $\mathcal{P} \times \mathbf{R}$ endowed with the *pre-symplectic* 2-form (i.e. closed 2-form of maximal rank):

$$\omega_H := \mathrm{d}p_k \wedge \mathrm{d}q^k - \mathrm{d}H \wedge \mathrm{d}t \tag{46}$$

(its "potential" $p_k\, \mathrm{d}q^k - H\, \mathrm{d}t$ is called the *Poincaré–Cartan invariant*). Obviously, ω_H is degenerate on $\mathcal{P} \times \mathbf{R}$ and the one-dimensional characteristic bundle of ω_H consists of the integral curves of the system in $\mathcal{P} \times \mathbf{R}$ (they may be considered as generated by the "super-Hamiltonian" which vanishes identically on the evolution space). This description may be called the "Heisenberg picture" of classical mechanics: physical states are not points in $\mathcal{P}$ but "particle's histories" in $\mathcal{P} \times \mathbf{R}$ (see [5]).

Let us construct the Hamiltonian structure for the theory based on our second-order Lagrangian $\mathcal{L}$. Let $\mathcal{P}$ denote the space of Hamiltonian variables, i.e. $(\mathbf{q}, \mathbf{p}, \mathbf{v}, \boldsymbol{\pi})$, where $\mathbf{p}$ and $\boldsymbol{\pi}$ stand for the momenta canonically conjugate to $\mathbf{q}$ and $\mathbf{v}$, respectively. Since our system is manifestly time-dependent (via the time dependence of the external field) we pass to the evolution space endowed with the pre-symplectic 2-form

$$\Omega_{\mathcal{H}} := \mathrm{d}p_k \wedge \mathrm{d}q^k + \mathrm{d}\pi_k \wedge \mathrm{d}v^k - \mathrm{d}\mathcal{H} \wedge \mathrm{d}t, \tag{47}$$

where $\mathcal{H}$ denotes the time-dependent particle's Hamiltonian.

To find $\mathcal{H}$ on $\mathcal{P} \times \mathbf{R}$ one has to perform the (time-dependent) Legendre transformation $(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}, \dot{\mathbf{v}}) \to (\mathbf{q}, \mathbf{p}, \mathbf{v}, \boldsymbol{\pi})$, i.e. one has to calculate $\dot{\mathbf{q}}$ and $\dot{\mathbf{v}}$ in terms of Hamiltonian variables, using formulae

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \dot{\pi}_k, \qquad \pi_k = \frac{\partial \mathcal{L}}{\partial \dot{v}^k}\,. \tag{48}$$

This transformation is singular due to linear dependence of $\mathcal{L}$ on $\dot{\mathbf{v}}$ and gives rise to the time-dependent constraints, given by Eqs. (25) and (40). The constraints can be easily solved, i.e. momenta p_k and π_k can be uniquely parametrized by the particle's position q^k, velocity v^k and the time t. Let $\mathcal{P}^*$ denote the constrained submanifold of the evolution space $\mathcal{P} \times \mathbf{R}$ parametrized by $(\mathbf{q}, \mathbf{p}^{\mathrm{kin}}, t)$. The reduced Hamiltonian on $\mathcal{P}^*$ reads.

$$\mathcal{H}(t, \mathbf{q}, \mathbf{v}) = p_k v^k + \pi_k \dot{v}^k - \mathcal{L} = \frac{m}{\sqrt{1 - \mathbf{v}^2}} + v^k p_k^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}). \tag{49}$$

Due to identity $u^\mu p_\mu^{\mathrm{int}} = 0$ (cf. [1]) we have

$$\mathcal{H}(t, \mathbf{q}, \mathbf{v}) = \frac{m}{\sqrt{1 - \mathbf{v}^2}} - p_0^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}). \tag{50}$$

Hence, we obtain:

Theorem 6. *The particle's Hamiltonian equals the "$-p_0$" component of the following gauge-invariant, four-vector*

$$p_\mu := p_\mu^{\mathrm{kin}} + p_\mu^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}) = m u_\mu + p_\mu^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}). \tag{51}$$

Using the laboratory-frame components of the external electromagnetic field we get

$$p_0^{\mathrm{int}}(t, \mathbf{q}, \mathbf{v}) = -\int \mathrm{d}^3x\, (\mathbf{D}\mathbf{D}_0 + \mathbf{B}\mathbf{B}_0). \tag{52}$$

Now, let us reduce the pre-symplectic 2-form (47) on $\mathcal{P}^*$. Calculating $p_k = p_k(\mathbf{q}, \mathbf{p}^{\mathrm{kin}}, t)$ and $\pi_k = \pi_k(\mathbf{q}, \mathbf{p}^{\mathrm{kin}}, t)$ from (25)–(40) and inserting them into (47) one obtains after a simple algebra:

$$\Omega_N = \mathrm{d}p_\mu^{\mathrm{kin}} \wedge \mathrm{d}q^\mu - e\, h_{\mu\nu}\, \mathrm{d}q^\mu \wedge \mathrm{d}q^\nu, \tag{53}$$

where $q^0 \equiv t$ and $h_{\mu\nu}$ is the following four-dimensional tensor:

$$e\, h_{\mu\nu}(t, \mathbf{q}, \mathbf{v}) := \frac{\partial p_\nu^{\rm int}}{\partial q^\mu} - \frac{\partial p_\mu^{\rm int}}{\partial q^\nu} \,. \tag{54}$$

Using techniques presented in Appendix A one easily proves:

Lemma 7.

$$\frac{\partial p_\nu^{\rm int}}{\partial q^\mu} = e\Pi_\mu^{\ \lambda} f_{\lambda\nu} \,, \tag{55}$$

where by

$$\Pi_\mu^{\ \lambda} := \delta_\mu^{\ \lambda} + u_\mu u^\lambda \tag{56}$$

we denote the projector on the hyperplane orthogonal to u^μ (i.e. to the particle's rest-frame hyperplane, see Appendix A). Therefore

$$h_{\mu\nu} = \Pi_\mu^{\ \lambda} f_{\lambda\nu} - \Pi_\nu^{\ \lambda} f_{\lambda\mu} = 2(f_{\mu\nu} - u_{[\mu} f_{\nu]\lambda} u^\lambda), \tag{57}$$

where $a_{[\alpha} b_{\beta]} := \frac{1}{2}(a_\alpha b_\beta - a_\beta b_\alpha)$. The form Ω_N is defined on a submanifold of cotangent bundle T^*M defined by the particle's "mass shell" $(p^{\rm kin})^2 = -m^2$.

Observe that the 2-form (53) has the same structure as Souriau's 2-form (7). They differ by the "curvature" 2-forms f and h only. However, the difference "$h - f$" vanishes identically along the particle's trajectories due to the fact that both $f_{\mu\nu}$ and $h_{\mu\nu}$ have the same projections in the direction of u^μ (see formula (10)). We conclude that the characteristic bundle of Ω_N and Ω_S are the same and they are described by the following equations:

$$\dot{q}^k = v^k, \tag{58}$$

$$\dot{v}^k = \sqrt{1 - \mathbf{v}^2}\, \frac{e}{m}(g^{kl} - v^k v^l)(E_l + \epsilon_{lij} v^i B^j), \tag{59}$$

which are equivalent to the Lorentz equations (45). From the physical point of view these forms are completely equivalent.

Appendix A

Due to the complicated dependence of the Coulomb field $\mathbf{D}_0$ and $\mathbf{B}_0$ on the particle's position $\mathbf{q}$ and velocity $\mathbf{v}$, formulae containing the respective derivatives of these fields are rather complex. To simplify the proofs, we shall use for calculations the particle's rest-frame, instead of the laboratory frame. The frame associated with a particle moving along a trajectory ζ may be defined as follows (cf. [1,3]): at each point $(t, \mathbf{q}(t)) \in \zeta$ we take the three-dimensional hyperplane Σ_t orthogonal to the four-velocity u^μ (the *rest-frame hypersurface*). We parametrize Σ_t by Cartesian coordinates (x^k), $k = 1, 2, 3$, centred at the particle's position (i.e. the point $x^k = 0$ belongs always to ζ). Obviously, there are infinitely many such coordinate systems on Σ_t, which differ from each other by an O(3)-rotation.

To fix uniquely coordinates (x^k), we choose the unique boost transformation relating the laboratory time axis $\partial/\partial y^0$ with the four-velocity vector $U := u^\mu \partial/\partial y^\mu$. Next, we define the position of the $\partial/\partial x^k$-axis on Σ_t by transforming the corresponding $\partial/\partial y^k$-axis of the laboratory frame by the same boost. The final formula relating Minkowskian coordinates (y^μ) with the new parameters (t, x^k) may be easily calculated (see e.g. [3]) from the above definition:

$$y^0(t, x^l) := t + \frac{1}{\sqrt{1-\mathbf{v}^2(t)}}\, x^l v_l(t),$$
$$y^k(t, x^l) := q^k(t) + (\delta^k_l + \varphi(\mathbf{v}^2) v^k v_l) x^l, \tag{A.1}$$

where we denote $\varphi(z) := (1/z)(1/\sqrt{1-z} - 1) = 1/\sqrt{1-z}(1+\sqrt{1-z})$.

Observe that the particle's Coulomb field has in this co-moving frame extremely simple form:

$$\mathcal{D}_0(\mathbf{x}) = \frac{e\mathbf{x}}{4\pi r^3}, \qquad \mathcal{B}_0(\mathbf{x}) = 0, \tag{A.2}$$

where $r := |\mathbf{x}|$. That is why the calculations in this frame are much easier than in the laboratory one.

Let $\mathcal{D}_k$ and $\mathcal{B}_k$ denote the rest-frame components of the electric and magnetic field. They are related to D_k and B_k as follows:

$$\mathcal{D}_k(\mathbf{x}, t; \mathbf{q}, \mathbf{v}) = \frac{1}{\sqrt{1-\mathbf{v}^2}}[(\delta^l_k - \sqrt{1-\mathbf{v}^2}\varphi(\mathbf{v}^2) v^l v_k) D_l(y) - \epsilon_{kij} v^i B^j(y)], \tag{A.3}$$
$$\mathcal{B}_k(\mathbf{x}, t; \mathbf{q}, \mathbf{v}) = \frac{1}{\sqrt{1-\mathbf{v}^2}}[(\delta^l_k - \sqrt{1-\mathbf{v}^2}\varphi(\mathbf{v}^2) v^l v_k) B_l(y) + \epsilon_{kij} v^i D^j(y)], \tag{A.4}$$

(the matrix $(\delta^l_k - \sqrt{1-\mathbf{v}^2}\,\varphi(\mathbf{v}^2) v^l v_k)$ comes from the boost transformation).

The field evolution with respect to the above *non-inertial* frame is a superposition of the following three transformations (cf. [1–3]):

- time-translation in the direction of U,
- boost in the direction of the particle's acceleration a^k,
- purely spatial O(3)-rotation around the vector ω_m,

where

$$a^k := \frac{1}{1-\mathbf{v}^2}(\delta^k_l + \varphi(\mathbf{v}^2) v^k v_l)\dot{v}^l, \tag{A.5}$$
$$\omega_m := \frac{1}{\sqrt{1-\mathbf{v}^2}}\,\varphi(\mathbf{v}^2) v^k \dot{v}^l \epsilon_{klm}. \tag{A.6}$$

Therefore, the Maxwell equations read (cf. [2,3]):

$$\dot{\mathcal{D}}^n = \sqrt{1-\mathbf{v}^2}\,\frac{\partial}{\partial x^m}[(\epsilon^{mk}{}_i \mathcal{D}^n - \epsilon^{nk}{}_i \mathcal{D}^m)\omega_k x^i - \epsilon^{mn}{}_k (1 + a^i x_i)\mathcal{B}^k] \tag{A.7}$$
$$\dot{\mathcal{B}}^n = \sqrt{1-\mathbf{v}^2}\,\frac{\partial}{\partial x^m}[(\epsilon^{mk}{}_i \mathcal{B}^n - \epsilon^{nk}{}_i \mathcal{B}^m)\omega_k x^i + \epsilon^{mn}{}_k (1 + a^i x_i)\mathcal{D}^k], \tag{A.8}$$

(the factor $\sqrt{1-\mathbf{v}^2}$ is necessary, because the time t, which we used to parametrize the particle's trajectory, is not a proper time along ζ but the laboratory time).

On the other hand, the time derivative with respect to the co-moving frame may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v^k \frac{\partial}{\partial q^k} + \dot{v}^k \frac{\partial}{\partial v^k} = \left(\frac{\partial}{\partial t}\right)_U + \dot{v}^k \frac{\partial}{\partial v^k} . \tag{A.9}$$

Therefore, taking into account (A.7) and (A.8) we obtain

$$\left(\frac{\partial}{\partial t}\right)_U \mathcal{D}^n = \sqrt{1-\mathbf{v}^2}\, \epsilon^{nmk} \partial_m \mathcal{B}_k , \tag{A.10}$$

$$\left(\frac{\partial}{\partial t}\right)_U \mathcal{B}^n = -\sqrt{1-\mathbf{v}^2}\, \epsilon^{nmk} \partial_m \mathcal{D}_k , \tag{A.11}$$

and

$$\frac{\partial}{\partial \dot{v}^l} \mathcal{D}^n = \sqrt{1-\mathbf{v}^2}\, \partial_m \left[\frac{\partial \omega_k}{\partial \dot{v}^l} \left(\epsilon_i^{mk} \mathcal{D}^n - \epsilon_i^{nk} \mathcal{D}^m \right) x^i - \frac{\partial a^k}{\partial \dot{v}^l} \epsilon_k^{mn} x_i \mathcal{B}^k \right] , \tag{A.12}$$

$$\frac{\partial}{\partial \dot{v}^l} \mathcal{B}^n = \sqrt{1-\mathbf{v}^2}\, \partial_m \left[\frac{\partial \omega_k}{\partial \dot{v}^l} \left(\epsilon_i^{mk} \mathcal{D}^n - \epsilon_i^{nk} \mathcal{D}^m \right) x^i + \frac{\partial a^k}{\partial \dot{v}^l} \epsilon_k^{mn} x_i \mathcal{B}^k \right] . \tag{A.13}$$

To calculate the derivatives of $\mathcal{D}^k$ and $\mathcal{B}^k$ with respect to the particle's position observe that

$$\frac{\partial}{\partial y^k} = -\frac{v_k}{\sqrt{1-\mathbf{v}^2}} U + (\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k) \frac{\partial}{\partial x^i} . \tag{A.14}$$

Therefore

$$\frac{\partial}{\partial q^k} \mathcal{D}^n = -\frac{v_k}{\sqrt{1-\mathbf{v}^2}} \epsilon^{nmi} \partial_m \mathcal{B}_i + (\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k) \partial_i \mathcal{D}^n , \tag{A.15}$$

$$\frac{\partial}{\partial q^k} \mathcal{B}^n = \frac{v_k}{\sqrt{1-\mathbf{v}^2}} \epsilon^{nmi} \partial_m \mathcal{D}_i + (\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k) \partial_i \mathcal{B}^n . \tag{A.16}$$

Now, using (A.10)–(A.13) and (A.15) and (A.16) we prove Lemmas 2–5.

A.1. Proof of Lemma 2

Observe that "interaction static moment" (29) in the particle's rest-frame reads:

$$R_k^{\text{int}} := \int_{\Sigma_t} x_k (\mathcal{D}_0 \mathcal{D} + \mathcal{B}_0 \mathcal{B})\, \mathrm{d}^3 x = \frac{e}{4\pi} \int_{\Sigma_t} \frac{x_k x^i}{r^3} \mathcal{D}^i\, \mathrm{d}^3 x . \tag{A.17}$$

Taking into account that

$$r_k^{\text{int}} = \frac{1}{\sqrt{1-\mathbf{v}^2}} \left(\delta_k^i - \sqrt{1-\mathbf{v}^2}\, \varphi(\mathbf{v}^2) v^i v_k \right) R_i^{\text{int}} , \tag{A.18}$$

we obtain the formula for π_k in terms of R_i^{int}:

$$\pi_k = -\frac{1}{\sqrt{1-\mathbf{v}^2}}(\delta_k^i + \varphi(\mathbf{v}^2)v^i v_k)R_i^{\text{int}}. \tag{A.19}$$

Now, using (A.12) one gets

$$\frac{\partial}{\partial v^l} R_i^{\text{int}} = \sqrt{1-\mathbf{v}^2}\left\{\frac{\partial a^m}{\partial \dot{v}^l}\mathbf{X}_{im} - \frac{\partial \omega^m}{\partial \dot{v}^l}\epsilon^j_{\ im}R_j^{\text{int}}\right\}, \tag{A.20}$$

where

$$\mathbf{X}_{im} = \frac{e}{4\pi}\epsilon_{ijk}\int_{\Sigma_t}\frac{x^j x_m}{r^3}\mathcal{B}^k\,\mathrm{d}^3x. \tag{A.21}$$

Therefore

$$\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} = A^i_{\ kl}R_i^{\text{int}} - B_{kl}^{im}\mathbf{X}_{im}, \tag{A.22}$$

where

$$\begin{aligned} A^i_{\ kl} = &\frac{\partial}{\partial v^k}\left[\frac{1}{\sqrt{1-\mathbf{v}^2}}(\delta_l^i + \varphi(\mathbf{v}^2)v^i v_l)\right] \\ &-\frac{\partial}{\partial v^l}\left[\frac{1}{\sqrt{1-\mathbf{v}^2}}(\delta_k^i + \varphi(\mathbf{v}^2)v^i v_k)\right] \\ &+\epsilon^i_{\ jm}\left[(\delta_k^j + \varphi(\mathbf{v}^2)v^j v_k)\frac{\partial \omega^m}{\partial \dot{v}^l} - (\delta_l^j + \varphi(\mathbf{v}^2)v^j v_l)\frac{\partial \omega^m}{\partial \dot{v}^k}\right], \end{aligned} \tag{A.23}$$

$$\begin{aligned} B_{kl}^{im} &= (\delta_k^i + \varphi(\mathbf{v}^2)v^i v_k)\frac{\partial a^m}{\partial \dot{v}^l} - (\delta_l^i + \varphi(\mathbf{v}^2)v^i v_l)\frac{\partial a^m}{\partial \dot{v}^k} \\ &= (1-\mathbf{v}^2)\left(\frac{\partial a^i}{\partial \dot{v}^k}\frac{\partial a^m}{\partial \dot{v}^l} - \frac{\partial a^i}{\partial \dot{v}^l}\frac{\partial a^m}{\partial \dot{v}^k}\right). \end{aligned} \tag{A.24}$$

Using the following properties of the function $\varphi(z)$:

$$2\varphi(z) - (1-z)^{-1} + z\varphi^2(z) = 0, \tag{A.25}$$

$$2\varphi'(z) - (1-z)^{-1}\varphi(z) - \varphi^2(z) = 0, \tag{A.26}$$

one easily shows that $A^i_{\ kl} \equiv 0$. Moreover, observe that B_{kl}^{im} defined in (A.24) is antisymmetric in (im). Therefore, to prove (36) it is sufficient to show that the quantity $\mathbf{X}_{im}$ is symmetric in (im). Taking into account that $\mathcal{B}^k = \epsilon^{klm}\partial_l\mathcal{A}_m$, where $\mathcal{A}_m$ stands for the rest-frame components of vector potential, one immediately gets

$$\epsilon_{ijk}\int_{\Sigma_t}\frac{x^j x_m}{r^3}\mathcal{B}^k\,\mathrm{d}^3x = \int_{\Sigma_t} r^{-5}(\mathcal{A}_k x^k)(3x_i x_m - r^2 g_{im})\,\mathrm{d}^3x, \tag{A.27}$$

which ends the proof of (36).

A.2. Proof of Lemma 3

To prove (37) observe that

$$\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} = \left(\frac{\partial}{\partial t}\right)_U \pi_k = -\frac{1}{\sqrt{1-\mathbf{v}^2}}(\delta^i_k + \varphi(\mathbf{v}^2) v^i v_k) \left(\frac{\partial}{\partial t}\right)_U R^{\rm int}_i. \tag{A.28}$$

Now, using (A.10) we obtain

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_U R^{\rm int}_i &= \sqrt{1-\mathbf{v}^2} \int_{\Sigma_t} \frac{x_i x_k}{r^3} \epsilon^{kjm} \partial_j \mathcal{B}_m \, \mathrm{d}^3 x \\ &= \sqrt{1-\mathbf{v}^2} \int_{\Sigma_t} \partial_j \left(\frac{x_i x_k}{r^3} \epsilon^{kjm} \mathcal{B}_m \right) \mathrm{d}^3 x \\ &\quad + \sqrt{1-\mathbf{v}^2} \int_{\Sigma_t} \epsilon_i{}^{km} \frac{x_k}{r^3} \mathcal{B}_m \, \mathrm{d}^3 x. \end{aligned} \tag{A.29}$$

Due to the Gauss theorem

$$\int_{\Sigma_t} \partial_j \left(\frac{x_i x_k}{r^3} \epsilon^{kjm} \mathcal{B}_m \right) \mathrm{d}^3 x = \int_{\partial \Sigma_t} \frac{x_j x_i x_k}{r^4} \epsilon^{kjm} \mathcal{B}_m \, \mathrm{d}\sigma \equiv 0, \tag{A.30}$$

where $\mathrm{d}\sigma$ denotes the surface measure on $\partial \Sigma_t$. Moreover, observe that "interaction momentum" in the particle's rest-frame reads

$$P^{\rm int}_i := \epsilon_{ikm} \int_{\Sigma_t} (\mathcal{D}^k_0 \mathcal{B}^m + \mathcal{D}^k \mathcal{B}^m_0) \, \mathrm{d}^3 x = \frac{e}{4\pi} \epsilon_{ikm} \int_{\Sigma_t} \frac{x^k}{r^3} \mathcal{B}^m \, \mathrm{d}^3 x. \tag{A.31}$$

Therefore

$$\left(\frac{\partial}{\partial t}\right)_U R^{\rm int}_i = \sqrt{1-\mathbf{v}^2} \, P^{\rm int}_i. \tag{A.32}$$

Using the relation between $p^{\rm int}_k$ and $P^{\rm int}_i$

$$p^{\rm int}_k = (\delta^i_k + \varphi(\mathbf{v}^2) v^i v_k) P^{\rm int}_i \tag{A.33}$$

we finally get (37).

A.3. Proof of Lemma 4

Using (A.13) and (A.15) we obtain

$$\frac{\partial}{\partial v^l} P^{\rm int}_i = \sqrt{1-\mathbf{v}^2} \left\{ \frac{\partial \omega^m}{\partial \dot{v}^l} \epsilon^j_{im} P^{\rm int}_j - \frac{\partial a^m}{\partial \dot{v}^l} \mathbf{Y}_{mi} \right\}, \tag{A.34}$$

$$\frac{\partial}{\partial v^l} R^{\rm int}_i = \frac{v_l}{\sqrt{1-\mathbf{v}^2}} P^{\rm int}_i + (\delta^j_l + \varphi(\mathbf{v}^2) v^j v_l) \mathbf{Y}_{ij}. \tag{A.35}$$

where

$$\mathbf{Y}_{ij} := \frac{e}{4\pi} \int_{\Sigma_t} \frac{x_i}{r^5}(3x^k x_j - r^2 \delta_j^k)\mathcal{D}_k \, \mathrm{d}^3x. \tag{A.36}$$

Now, taking into account (A.19) and (A.33) we have

$$\frac{\partial p_k^{\text{int}}}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k} = C^i{}_{kl} P_i^{\text{int}}, \tag{A.37}$$

where

$$C^i{}_{kl} := \frac{\partial}{\partial v^l}(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k) - \sqrt{1-\mathbf{v}^2}(\delta_k^j + \varphi(\mathbf{v}^2) v^j v_k)\frac{\partial \omega^m}{\partial v^l} \epsilon^i{}_{jm}$$

$$+ \frac{v_k}{1-\mathbf{v}^2}(\delta_l^i + \varphi(\mathbf{v}^2) v^i v_l). \tag{A.38}$$

One easily shows that due to properties (A.25) and (A.26) $C^i_{kl} \equiv 0$, which ends the proof of (43).

A.4. *Proof of Lemma 5*

Finally, to prove (44) let us observe that

$$\frac{\partial p_k^{\text{int}}}{\partial t} + v^l \frac{\partial p_k^{\text{int}}}{\partial q^l} = \left(\frac{\partial}{\partial t}\right)_U p_k^{\text{int}} = \left(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k\right)\left(\frac{\partial}{\partial t}\right)_U P_i^{\text{int}}. \tag{A.39}$$

Now, due to (A.11) we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t}\right)_U P_i^{\text{int}} &= \sqrt{1-\mathbf{v}^2}\, \frac{e}{4\pi} \int_{\partial \Sigma_t} \frac{1}{r^2}\mathcal{D}_i \, \mathrm{d}\sigma \\
&= -\sqrt{1-\mathbf{v}^2}\, \frac{e}{4\pi} \lim_{r_0 \to 0} \int_{S(r_0)} \frac{1}{r^2}\mathcal{D}_i \, \mathrm{d}\sigma \\
&= -\sqrt{1-\mathbf{v}^2}\, e\mathcal{D}_i(t, 0)\,,
\end{aligned} \tag{A.40}$$

where we choose as two pieces of a boundary $\partial \Sigma_t$ a sphere at infinity and a sphere $S(r_0)$. Using the fact that in the Heaviside–Lorentz system of units $\mathcal{D}_k = \mathcal{E}_k$ and taking into account the formula (A.3) we finally obtain (44).

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